

# DERIVED CATEGORIES OF BURNIAT SURFACES AND EXCEPTIONAL COLLECTIONS

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**ABSTRACT.** We construct an exceptional collection  $\Upsilon$  of maximal possible length 6 on any of the Burniat surfaces with  $K_X^2 = 6$ , a 4-dimensional family of surfaces of general type with  $p_g = q = 0$ . We also calculate the DG algebra of endomorphisms of this collection and show that the subcategory generated by this collection is the same for all Burniat surfaces. The semiorthogonal complement  $\mathcal{A}$  of  $\Upsilon$  is an “almost phantom” category: it has trivial Hochschild homology, and  $K_0(\mathcal{A}) = \mathbb{Z}_2^6$ .

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## 1. Introduction

In a surprising recent paper [BGvBS12], Böhning, Graf von Bothmer, and Sosna produced an exceptional sequence of maximal possible length 11 on the classical Godeaux surface, which is the  $\mathbb{Z}_5$ -quotient of the Fermat quintic surface in  $\mathbb{P}^3$ . The computation is quite involved and is heavily computer-aided. It uses the  $E_8$  root system and a very careful study of effective curves on the Godeaux surface.

In this paper we make a similar but much easier computation for Burniat surfaces, which can be described either as Galois  $\mathbb{Z}_2^2$ -covers of  $\mathrm{Bl}_3 \mathbb{P}^2$  or as  $\mathbb{Z}_2^3$ -quotients of  $(2, 2, 2)$ -divisors in a product of three elliptic curves.

The Godeaux surface has the same Picard lattice as a del Pezzo surface of degree 1. The Picard lattice of a Burniat surface is isomorphic to that of a del Pezzo surface of degree 6. So, essentially the  $E_8$  lattice of the classical Godeaux surface is replaced by a much smaller lattice  $E_3 = A_2 \times A_1$  with a very small Weil group  $S_3 \times S_2$ . The Picard number of a Burniat surface is 4, and a maximal exceptional sequence has length only 6.

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Note that the results of [BGvBS12] apply to a unique surface, and the results of our computations apply to all Burniat surfaces which form a 4-dimensional family. For each of these surfaces  $X$ , we find an exceptional collection  $\Upsilon = (L_1, \dots, L_6)$  of length 6 consisting of line bundles  $L_i$ . This collection splits into 3 blocks of sizes  $2 + 3 + 1$ , and the sheaves in the same block are mutually orthogonal.

The collection  $\Upsilon$  gives a semiorthogonal decomposition for the bounded derived category of coherent sheaves on  $X$  of the form

$$\mathbf{D}^b(\mathrm{coh}(X)) = \langle L_1, \dots, L_6, \mathcal{A} \rangle$$

The admissible subcategory  $\mathcal{A}$  is “almost phantom”: it has trivial Hochschild homology  $\mathrm{HH}_*(\mathcal{A}) = 0$  and its Grothendieck group  $K_0(\mathcal{A})$  is only the torsion group  $\mathbb{Z}_2^6$ .

We also calculate the differential graded (DG) algebra of endomorphisms of the collection  $\Upsilon = (L_1, \dots, L_6)$  and show that it is formal, i.e. it is quasi-isomorphic to its cohomology algebra. This algebra is constant in the family, it is the same for any Burniat surface.

On the other hand, it is well known (see [BO01]) that a smooth variety  $X$  with ample canonical class can be uniquely reconstructed from the derived category  $\mathbf{D}^b(\mathrm{coh}(X))$ . This means that quite surprisingly, in spite of  $\mathcal{A}$  being “almost phantom”, all non-trivial variations of a Burniat surface in the 4-dimensional family are hidden away in the subcategory  $\mathcal{A}$  and a gluing functor between this subcategory and a fixed subcategory  $\mathcal{D}$ .

It would be very interesting to understand if there exist admissible subcategories in the bounded derived categories of smooth varieties for which not only Hochschild homology but also the Grothendieck group  $K_0$  is trivial (“phantom” categories.) Arguments for and against existence of “phantom” and “almost phantom” categories were previously discussed in the literature (see e.g. [Kuz09]) and experts’ opinions on this vary. One candidate for finding a “phantom” is Barlow surface. See the end of [DKK12] for a related conjecture.

Throughout the paper, we work over an algebraically closed field  $k$  of characteristic different from 2. The only point where characteristic is possibly important is the moduli of Burniat surfaces. For any  $k$  with  $\mathrm{char} k \neq 2$ , there is a 4-dimensional family coming from line arrangements. Over  $\mathbb{C}$ , all Burniat surfaces with  $K_X^2 = 6$  are in this family, see the discussion on page 4.

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## 2. Curves on Burniat surfaces

Burniat surfaces are surfaces of general type with  $p_g = q = 0$  which were introduced in [Bur66] (see also [Pet77]) and from a different point of view by Inoue [Ino94]. Burniat surfaces come in several deformation families with  $2 \leq K_X^2 \leq 6$ . We will consider the unique family with  $K_X^2 = 6$ , which is sometimes called “primary” Burniat surfaces. For a detailed study of Burniat surfaces, including the proof of the fact that they are Inoue surfaces, see [BC11]. Some basic facts about Burniat-Inoue surfaces can also be found in [BHPVdV04, VII.11].

The easiest way to describe Burniat surfaces with  $K_X^2 = 6$  is as Galois  $\mathbb{Z}_2^2$ -covers of the blowup  $\text{Bl}_3 \mathbb{P}^2$  of  $\mathbb{P}^2$  at three points, a toric del Pezzo surface of degree 6.

Recall from [Par91] that a  $\mathbb{Z}_2^2$ -cover  $\pi: X \rightarrow Y$  with smooth and projective  $X, Y$  is determined by three branch divisors  $\bar{A}, \bar{B}, \bar{C}$  and three invertible sheaves  $L_1, L_2, L_3$  on the base  $Y$  satisfying fundamental relations  $L_2 \otimes L_3 \simeq L_1(\bar{A})$ ,  $L_3 \otimes L_1 \simeq L_2(\bar{B})$ ,  $L_1 \otimes L_2 \simeq L_3(\bar{C})$ . These relations imply that  $L_1^2 \simeq \mathcal{O}_Y(\bar{B} + \bar{C})$ ,  $L_2^2 \simeq \mathcal{O}_Y(\bar{C} + \bar{A})$ ,  $L_3^2 \simeq \mathcal{O}_Y(\bar{A} + \bar{B})$ .

One has  $X = \text{Spec}_Y \mathcal{A}$ , where the  $\mathcal{O}_Y$ -algebra  $\mathcal{A}$  is  $\mathcal{O}_Y \oplus \bigoplus_{i=1}^3 L_i^{-1}$ . The multiplication is determined by three sections in

$$\text{Hom}(L_i^{-1} \otimes L_j^{-1}, L_k^{-1}) = H^0(L_i \otimes L_j \otimes L_k^{-1}),$$

where  $\{i, j, k\}$  is a permutation of  $\{1, 2, 3\}$ , i.e. by sections of the sheaves  $\mathcal{O}_Y(\bar{A})$ ,  $\mathcal{O}_Y(\bar{B})$ ,  $\mathcal{O}_Y(\bar{C})$  vanishing on  $\bar{A}$ ,  $\bar{B}$ ,  $\bar{C}$ .

In our case, the divisors  $\bar{A} = \sum_{i=0}^3 \bar{A}_i$ ,  $\bar{B} = \sum_{i=0}^3 \bar{B}_i$ ,  $\bar{C} = \sum_{i=0}^3 \bar{C}_i$  are the ones shown in red, blue, and black in the central picture of Figure 1 below. The del Pezzo surface has two different contractions to  $\mathbb{P}^2$  related by a quadratic transformation, and the images of the divisors form a special line configuration on either  $\mathbb{P}^2$ . We denote by  $|h_1|$ , resp.  $|h_2|$  the linear systems contracting  $\text{Bl}_3 \mathbb{P}^2$  to the left, resp. the right  $\mathbb{P}^2$ .

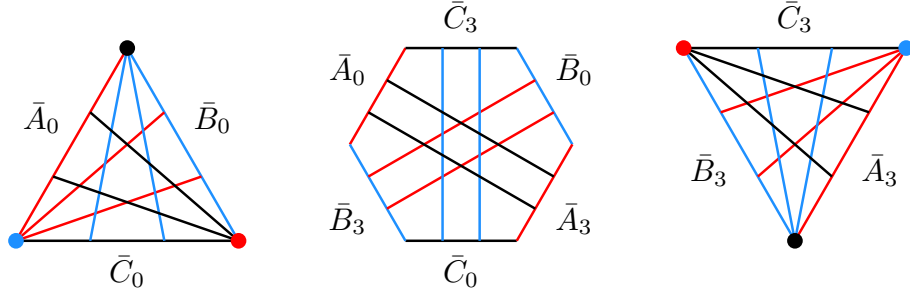


FIGURE 1. Burniat configuration on  $\text{Bl}_3 \mathbb{P}^2$

The curves  $\bar{A}_0$  and  $\bar{A}_3$  are  $(-1)$ -curves. The curves  $\bar{A}_1$  and  $\bar{A}_2$  are rational curves with square 0. They are divisors in a pencil  $|f_1|$  which has two reducible fibers  $\bar{A}_0 + \bar{C}_3$  and  $\bar{C}_0 + \bar{A}_3$ . Similarly,  $\bar{B}_1, \bar{B}_2$  are divisors in a pencil  $|f_2|$  and  $\bar{C}_1, \bar{C}_2$  are divisors in a pencil  $|f_3|$ .

We recall the following well known facts about Burniat surfaces:

(1) The surface  $X$  is smooth iff the configuration of the branch curves is generic, i.e. the branch divisors do not share components, at any point no more than two intersect and they belong to different branch divisors.

(2) The map  $\pi$  is ramified, with index 2, over the curves  $\bar{A}_i, \bar{B}_i, \bar{C}_i$ . Let us denote the corresponding ramification curves on a Burniat surface by  $A_i, B_i, C_i$ . One has  $\pi^*(\bar{A}_i) = 2A_i$ , etc. For the canonical class, one has numerically

$$K_X = \pi^*(K_Y + \frac{1}{2} \sum (\bar{A}_i + \bar{B}_i + \bar{C}_i)) = \pi^*(-\frac{1}{2} K_Y).$$

Therefore,  $K_X$  is ample and  $K_X^2 = K_Y^2 = 6$ .

(3) One has  $h^i(\mathcal{O}_X) = h^i(\mathcal{O}_Y) = 0$  for  $i = 1, 2$ . Thus,  $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 1$ . Noether's formula  $\chi(\mathcal{O}) = (c_1^2 + c_2)/12$  implies that  $X$  and  $Y$  have the same Betti and Picard numbers. Hence,  $\text{Pic } Y \simeq \mathbb{Z}^4$  and  $\text{Pic } X/\text{Tors} \simeq \mathbb{Z}^4$ .

(4) The torsion subgroup of  $\text{Pic } X$  is isomorphic to  $\mathbb{Z}_2^6$ , see [Pet77].

(5) The fundamental group is an extension of  $\mathbb{Z}_2^3$  by  $\mathbb{Z}^6$  and is not abelian. This follows from Inoue's construction of  $X$  as a free  $\mathbb{Z}_2^3$ -quotient of a divisor in the product of three elliptic curves, see [Ino94, p.315], [BC11].

(6) The Burniat configuration of branch curves on  $\text{Bl}_3 \mathbb{P}^2$  is uniquely determined by a line configuration in  $\mathbb{P}^2$ . That one is described by a 4-dimensional family. Indeed, for the configuration in, say, the left  $\mathbb{P}^2$ , one can fix  $\bar{A}_0, \bar{B}_0, \bar{C}_0$  and  $\bar{A}_1, \bar{B}_1$ ; this gives a unique line configuration with a trivial automorphism group. Then moving the other 4 lines  $\bar{A}_2, \bar{B}_2, \bar{C}_1, \bar{C}_2$  gives a 4-dimensional family.

Over  $\mathbb{C}$ , one knows from Mendez-Pardini [MLP01] that these are *all* the Burniat surfaces with  $K_X^2 = 6$ . They prove that a deformation of an abelian cover in this case is again an abelian cover. Since  $\mathbb{P}^2$  does not deform, all the deformations come from varying the curves, thus varying the lines in  $\mathbb{P}^2$ . The main result of [MLP01] is that over  $\mathbb{C}$  the Burniat surfaces presented above form a connected component in the moduli space of surfaces of general type.

Moreover, [AP09] describes the compactification of this 4-dimensional moduli space obtained by adding stable surfaces, i.e. surfaces with semi log canonical singularities and ample canonical class.

**Lemma 2.1.** *The homomorphism  $\bar{D} \mapsto \frac{1}{2}\pi^*(\bar{D})$  defines an isomorphism of integral lattices  $\frac{1}{2}\pi^*: \text{Pic } Y \rightarrow \text{Pic } X/\text{Tors}$ . One has  $\frac{1}{2}\pi^*(-K_Y) = K_X$ .*

*Proof.* Since  $\deg \pi = 4$ , one has  $\frac{1}{2}\pi^*(\bar{D}_1) \cdot \frac{1}{2}\pi^*(\bar{D}_2) = \frac{1}{4}\pi^*(\bar{D}_1 \bar{D}_2) = \bar{D}_1 \bar{D}_2$ . This defines an isomorphism  $\text{Pic } Y \otimes \mathbb{Q} \rightarrow \text{Pic } X \otimes \mathbb{Q}$  together with the intersection products. In fact, the image of  $\text{Pic } Y$  is integral. Indeed, the branch divisors  $\bar{A}_i, \bar{B}_i, \bar{C}_i$  generate  $\text{Pic } Y$  and for each of them  $\frac{1}{2}\pi^*(\bar{D})$  is an integral cycle. This shows that  $\frac{1}{2}\pi^*(\text{Pic } Y) \subset \text{Pic } X/\text{Tors}$  is a sublattice of finite index. Since the lattice  $\text{Pic } Y$  is unimodular, one must have the equality.  $\square$

The proof of the lemma shows that numerically we can identify the curves  $\bar{D} = \bar{A}_i, \bar{B}_i, \bar{C}_i$  downstairs with the corresponding curves  $D := \frac{1}{2}\pi^*(\bar{D})$  upstairs. For  $i = 0, 3$ , the curves  $A_i, B_i, C_i$  on  $X$  are elliptic curves with  $D^2 = -1$ . For  $i = 1, 2$ , they are genus 2 curves with  $D^2 = 0$ .

Every point of intersection  $\bar{P} = \bar{D}_1 \cap \bar{D}_2$  of these curves on  $Y$  gives *only one* point of intersection  $P = D_1 \cap D_2$  on  $X$  since  $D_1 D_2 = \bar{D}_1 \bar{D}_2 = 1$ . (Another way to see it: the divisors  $\bar{D}_1, \bar{D}_2$  belong to different branch divisors, so the cover  $\pi$  is fully ramified at  $P$ ). Thus, the configuration of the 12 curves and their intersections on  $X$  is exactly the same as for  $Y$ , and we can continue to use Figure 1 to visualize it.

**Notation 2.2.** We give the elliptic curve  $A_0$  on  $X$  a group structure by fixing the point  $B_3 \cap A_0$  as the origin. Note that the four points of intersection of  $A_0$  with the other branch divisors split as  $1 + 3$ : one point in  $A_0 \cap B$  and three points in  $A_0 \cap C$ . Our choice of the origin is determined by this splitting.

It is easy to see that the differences between the intersections of  $A_0$  with  $B$  and  $C$  are 2-torsion points. We fix an isomorphism of the 2-torsion group  $A_0[2] \rightarrow \mathbb{Z}_2^2$  by making the identifications

$$B_3 \cap A_0 = 00, \quad C_3 \cap A_0 = 10, \quad C_1 \cap A_0 = 01, \quad C_2 \cap A_0 = 11.$$

We make the same identification for the other 5 elliptic curves, rotating the hexagon in Figure 1 cyclically, so that  $C_2 \cap A_3 = 01$  in  $A_3[2]$ , and similarly for  $B_3, C_3$ .

The subgroup  $\mathbb{Z}.P_{00} + A_0[2] \subset \text{Pic } A_0$  is isomorphic to  $\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Every element of this subgroup can be written as a triple  $(a_0^0, a_0^1, a_0^2)$ . Similarly, we write elements in the groups  $\mathbb{Z}.P_{00} + B_0[2]$ ,  $\mathbb{Z}.P_{00} + C_0[2]$  as triples  $(b_0^0, b_0^1, b_0^2)$ ,  $(c_0^0, c_0^1, c_0^2)$ . We use similar notation for the three elliptic curves  $A_3, B_3, C_3$ .

We thank Rita Pardini for providing us with a proof of the following Lemma.

**Lemma 2.3.** *One has  $\mathcal{O}_{A_0}(K_X) = \mathcal{O}_{A_0}(-A_0) = \mathcal{O}_{A_0}(P_{00})$ , and similarly for the other 5 elliptic curves.*

*Proof.* Let  $q: W \rightarrow Y$  be an intermediate double cover corresponding to the branch divisor  $\bar{B} + \bar{C}$ , so that  $p: X \rightarrow W$  is the double cover for the branch divisor  $A' = q^{-1}(\bar{A})$ . Note that  $W$  is singular at the points above  $\bar{B} \cap \bar{C}$  but is smooth in a neighborhood of  $\bar{A}'$ , so those points can be disregarded in the computation.

For the double cover  $p$  and a connected component  $A_0$  of the branch divisor, it is immediate that  $\mathcal{O}_{A_0}(A')$  is  $p^*L|_{A_0}$ , where  $L$  on  $W$  is determined by the equality  $2L = A'$ . One computes that  $p^*L = \pi^*L_3 - p^*R_2 = \pi^*L_2 - p^*R_3$ , where  $R_2$  (resp.  $R_3$ ) is the preimage of  $B$  (resp. of  $C$ ) on  $W$ . Plugging this in gives  $\mathcal{O}_{A_0}(A_0) = \mathcal{O}_{A_0}(A') = \mathcal{O}_{A_0}(-P_{00})$ .  $\square$

**Theorem 2.4.** (1) *The homomorphism*

$$\begin{aligned} \phi: \text{Pic } X &\rightarrow \mathbb{Z} \times \text{Pic } A_0 \times \text{Pic } B_0 \times \text{Pic } C_0 \\ L &\mapsto (d(L) = L \cdot K_X, L|_{A_0}, L|_{B_0}, L|_{C_0}) \end{aligned}$$

*is injective, and the image is the subgroup of index 3 of*

$$\mathbb{Z} \times (\mathbb{Z}.P_{00} + A_0[2]) \times (\mathbb{Z}.P_{00} + B_0[2]) \times (\mathbb{Z}.P_{00} + C_0[2]) \simeq \mathbb{Z}^4 \times \mathbb{Z}_2^6$$

*consisting of the elements with  $d + a_0^0 + b_0^0 + c_0^0$  divisible by 3.*

(2)  *$\phi$  induces an isomorphism  $\text{Tors}(\text{Pic } X) \rightarrow A_0[2] \oplus B_0[2] \oplus C_0[2]$ .*

(3) *The curves  $A_i, B_i, C_i$ ,  $0 \leq i \leq 3$ , generate  $\text{Pic } X$ .*

*Proof.* Fix the contraction  $p: Y \rightarrow \mathbb{P}^2$  for which  $\bar{A}_0, \bar{B}_0, \bar{C}_0$  are the exceptional divisors, and let  $\bar{H}$  be the generator of  $\text{Pic } \mathbb{P}^2$ . Then  $-K_Y = 3\bar{H} - \bar{A}_0 - \bar{B}_0 - \bar{C}_0$ . Since the lattice  $\text{Pic } Y = \mathbb{Z}\bar{H} + \mathbb{Z}\bar{A}_0 + \mathbb{Z}\bar{B}_0 + \mathbb{Z}\bar{C}_0$  is unimodular, this implies that the homomorphism  $\bar{\phi}: \text{Pic } X / \text{Tors} = \text{Pic } Y \rightarrow \mathbb{Z}^4$  induced by  $\phi$  is injective and the image is the subgroup of index 3 consisting of the elements with  $d + a_0^0 + b_0^0 + c_0^0$  divisible by 3. This also implies that  $\text{Tors}(\text{Pic } X) = \ker \bar{\phi}$ .

In Table 1, we write down explicitly the restrictions of the curves  $A_i, B_i, C_i$  to  $\text{Pic } A_i, \text{Pic } B_i, \text{Pic } C_i$ . From this table, it is obvious that the homomorphism  $\text{Tors}(\text{Pic } X) \rightarrow A_0[2] \oplus B_0[2] \oplus C_0[2]$  is surjective. (Indeed,  $C_2 - C_1$  and  $C_2 - C_3 - B_0$  generate  $A_0[2]$ ; similarly for  $B_0[2]$ ,  $C_0[2]$ .) Since  $\text{Tors}(\text{Pic } X) \simeq \mathbb{Z}_2^6$  (see [Pet77] or [Ino94, p.315]), it must be an isomorphism.

The image of  $A_3$  in  $\text{Pic } Y$  is  $\bar{A}_3 = \bar{H} - \bar{B}_0 - \bar{C}_0$ . So, together with the curves  $A_0, B_0, C_0$  it generates  $\text{Pic } X / \text{Tors}$ , and all the curves together generate  $\text{Pic } X$ .  $\square$

**Lemma 2.5.** *The coordinates with respect to the triple  $(A_3, B_3, C_3)$  are related to the coordinates with respect to the triple  $(A_0, B_0, C_0)$  by the formulas*

$$3a_3^0 = d + a_0^0 - 2b_0^0 - 2c_0^0, \quad a_3^1 = a_0^1 + b_0^2 + (d + a_0^0 + b_0^0)(\text{mod } 2), \quad a_3^2 = a_0^2,$$

*and similarly for  $b_3, c_3$  rotating cyclically.*

	$d$	$a_0$	$b_0$	$c_0$	$a_3$	$b_3$	$c_3$
$A_0$	1	-1 00	0 00	0 00	0 00	1 10	1 00
$B_0$	1	0 00	-1 00	0 00	1 00	0 00	1 10
$C_0$	1	0 00	0 00	-1 00	1 10	1 00	0 00
$A_3$	1	0 00	1 10	1 00	-1 00	0 00	0 00
$B_3$	1	1 00	0 00	1 10	0 00	-1 00	0 00
$C_3$	1	1 10	1 00	0 00	0 00	0 00	-1 00
$A_1$	2	0 00	1 01	0 00	0 00	1 11	0 00
$A_2$	2	0 00	1 11	0 00	0 00	1 01	0 00
$B_1$	2	0 00	0 00	1 01	0 00	0 00	1 11
$B_2$	2	0 00	0 00	1 11	0 00	0 00	1 01
$C_1$	2	1 01	0 00	0 00	1 11	0 00	0 00
$C_2$	2	1 11	0 00	0 00	1 01	0 00	0 00
$K_X$	6	1 00	1 00	1 00	1 00	1 00	1 00

TABLE 1. Generators of  $\text{Pic } X$  and  $K_X$  in symmetric coordinates

*Proof.* The formula for  $a_3^0$  is easy. For  $a_3^1, a_3^2$ , the formulas come from putting Table 1 (mod 2), considered as a  $12 \times 19$  matrix with coefficients in  $\mathbb{Z}_2$ , in the reduced row-echelon form.  $\square$

### 3. Exceptional collections on $\text{Bl}_3 \mathbb{P}^2$

It is well-known that the bounded derived category of coherent sheaves  $\mathbf{D}^b(\text{coh}(S))$  on any del Pezzo surface  $S$  has a full exceptional collection ([Orl92], see also [KO94]). This is a particular case of a more general statement about derived categories of blowups.

First, recall the notion of an exceptional collection.

**Definition 3.1.** An object  $\mathcal{E}$  of a  $k$ -linear triangulated category  $\mathcal{D}$  is said to be exceptional if

$$\text{Hom}(\mathcal{E}, \mathcal{E}[m]) = \text{Ext}^m(\mathcal{E}, \mathcal{E}) = 0 \quad \text{for all } m \neq 0,$$

and  $\text{Hom}(\mathcal{E}, \mathcal{E}) = k$ . An ordered set of exceptional objects  $(\mathcal{E}_1, \dots, \mathcal{E}_n)$  is called an exceptional collection if  $\text{Hom}(\mathcal{E}_j, \mathcal{E}_i[m]) = 0$  for  $j > i$  and all  $m$ .

**Definition 3.2.** An exceptional collection  $(\mathcal{E}_1, \dots, \mathcal{E}_n)$  in a category  $\mathcal{D}$  is called full if it generates the category  $\mathcal{D}$ , i.e. the minimal full triangulated subcategory of  $\mathcal{D}$  containing all objects  $\mathcal{E}_i$  coincides with  $\mathcal{D}$ . In this case we say that  $\mathcal{D}$  has a semiorthogonal decomposition of the form

$$\mathcal{D} = \langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle.$$

**Definition 3.3.** The exceptional collection  $(\mathcal{E}_1, \dots, \mathcal{E}_n)$  is said to be strong if it satisfies the additional condition  $\text{Hom}(\mathcal{E}_j, \mathcal{E}_i[m]) = 0$  for all  $i, j$  and for  $m \neq 0$ .

The most studied example of an exceptional collection is the sequence of invertible sheaves  $\langle \mathcal{O}_{\mathbb{P}^n}, \dots, \mathcal{O}_{\mathbb{P}^n}(n) \rangle$  on the projective space  $\mathbb{P}^n$ . This exceptional collection is full and strong.

**Definition 3.4.** The algebra of a strong exceptional collection  $\Sigma = (\mathcal{E}_1, \dots, \mathcal{E}_n)$  is the algebra of endomorphisms  $B_\Sigma = \text{End}(\mathcal{T})$  of the object  $\mathcal{T} = \bigoplus_{i=1}^n \mathcal{E}_i$ .

Assume that the triangulated category  $\mathcal{D}$  has a full strong exceptional collection  $\Sigma = (\mathcal{E}_1, \dots, \mathcal{E}_n)$  and  $B_\Sigma$  is the corresponding algebra. Denote by  $\text{mod } -B_\Sigma$  the category of finite right modules over  $B_\Sigma$ . There is a theorem according to which if  $\mathcal{D}$  is an *enhanced triangulated category* in the sense of Bondal and Kapranov [BK90], then it is equivalent to the bounded derived category  $\mathbf{D}^b(\text{mod } -B_\Sigma)$ . This equivalence is given by the functor  $\mathbf{R}\text{Hom}(\mathcal{T}, -)$  (see [BK90]).

If a full exceptional collection  $\Sigma = (\mathcal{E}_1, \dots, \mathcal{E}_n)$  in an enhanced triangulated category  $\mathcal{D}$  is not strong, then we can consider a DG algebra (or  $A_\infty$ -algebra) of endomorphisms  $\mathcal{B}_\Sigma = \mathbf{R}\text{Hom}(\mathcal{T}, \mathcal{T})$  and the same functor will induce an equivalence between  $\mathcal{D}$  and the category of perfect objects  $\text{Perf}(\mathcal{B}_\Sigma)$  over  $\mathcal{B}_\Sigma$ .

For this fact and the main results on DG algebras and DG categories, we refer the reader to [Kel94, Kel06]. For the notions and techniques of  $A_\infty$ -algebras and  $A_\infty$ -categories, we refer to [Kel01, Sei08, Lef02].

Any full subcategory  $\mathcal{D} \cong \mathbf{D}^b(\text{coh}(Z))$  of the bounded derived category of coherent sheaves on a variety  $Z$  is enhanced. Assume that the subcategory  $\mathcal{D}$  is generated by an exceptional collection  $\Sigma = (\mathcal{E}_1, \dots, \mathcal{E}_n)$ . In this case we obtain an equivalence

$$\mathbf{R}\text{Hom}(\mathcal{T}, -) : \mathcal{D} \xrightarrow{\sim} \text{Perf}(\mathcal{B}_\Sigma).$$

If the exceptional collection  $\Sigma$  is full then  $\mathcal{D} \cong \mathbf{D}^b(\text{coh}(Z))$ , if the collection is strong then the DG algebra  $\mathcal{B}_\Sigma$  is quasi-isomorphic to the algebra  $B_\Sigma$  and  $\text{Perf}(\mathcal{B}_\Sigma) \cong \mathbf{D}^b(\text{mod } -B_\Sigma)$ .

**Theorem 3.5.** [Orl92, KO94] *Let  $p : S_K \rightarrow \mathbb{P}^2$  be a blowup of the projective plane  $\mathbb{P}^2$  at a set  $K = \{P_1, \dots, P_k\}$  of any  $k$  distinct points, and let  $E_1, \dots, E_k$  be the exceptional curves of the blowup. Then the sequence*

$$(1) \quad (\mathcal{O}_{S_K}, p^* \mathcal{O}_{\mathbb{P}^2}(1), p^* \mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{E_1}, \dots, \mathcal{O}_{E_k}),$$

*where  $\mathcal{O}_{E_i}$  are the structure sheaves of the exceptional  $-1$ -curves  $E_i$ , is a full strong exceptional collection on  $S_K$ . In particular, there is an equivalence*

$$(2) \quad \mathbf{D}^b(\text{coh}(S_K)) \cong \mathbf{D}^b(\text{mod } -B_K),$$

*where  $B_K$  is the algebra of homomorphisms of the exceptional collection (1).*

There are no restrictions on the set of points  $K = \{P_1, \dots, P_k\}$  in this theorem and, in particular, we do not need to assume that  $S_K$  is a del Pezzo surface.

Exceptional objects and exceptional collections on del Pezzo surfaces are well-studied objects. First, any exceptional object of the derived category is isomorphic to a sheaf up to translation. Second, any exceptional sheaf can be included in a full exceptional collection. Third, any full exceptional collection can be obtained from a given one by a sequence of natural operations on exceptional collections called *mutations*. All these facts can be found in the paper [KO94].

An exceptional collection is called a **block** if  $\mathcal{E}_i$  and  $\mathcal{E}_j$  are mutually orthogonal for all  $i \neq j$ , i.e.  $\text{Hom}(\mathcal{E}_j, \mathcal{E}_i[m]) = 0$  for all  $m$  and  $i \neq j$ . For example, the sheaves  $(\mathcal{O}_{E_1}, \dots, \mathcal{O}_{E_k})$  on  $S_K$  form a block.

A remarkable fact is that any del Pezzo surface  $S_K$  with  $3 \leq k \leq 8$  possesses a full exceptional collection consisting of three blocks (see [KN98]).

Consider the del Pezzo surface  $Y = \text{Bl}_3 \mathbb{P}^2$  that is a blow up of projective plane at three points. As in Section 2, we use  $f_1, f_2, f_3$  to denote the divisors defining the special pencils on  $Y$ , and  $h_1, h_2$  for the divisors defining the contractions  $Y \rightarrow \mathbb{P}^2$ .

**Theorem 3.6** ([KN98]). *The following collection on  $Y = \text{Bl}_3 \mathbb{P}^2$*

$$(3) \quad \Sigma = (\mathcal{O}_Y, \mathcal{O}_Y(f_1), \mathcal{O}_Y(f_2), \mathcal{O}_Y(f_3), \mathcal{O}_Y(h_1), \mathcal{O}_Y(h_2))$$

*is a full strong exceptional collection that is split into three blocks of sizes  $1+3+2$ , and the sheaves in the same block are mutually orthogonal.*

It is easy to see that this collection is strong and to calculate the algebra of endomorphisms  $B_\Sigma$  of this exceptional collection. There are only following nontrivial Hom's between objects of  $\Sigma$

$$\begin{aligned} \text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y(f_i)) &\cong k^2 && \text{for all } i = 1, 2, 3 \\ \text{Hom}(\mathcal{O}_Y, \mathcal{O}_Y(h_j)) &\cong k^3 && \text{for all } j = 1, 2 \\ \text{Hom}(\mathcal{O}_Y(f_i), \mathcal{O}_Y(h_j)) &\cong k && \text{for all } i = 1, 2, 3; j = 1, 2. \end{aligned}$$

With evident composition law they completely define the algebra  $B_\Sigma$  of endomorphisms of the collection  $\Sigma$ . Thus we obtain

**Proposition 3.7.** *There is an equivalence*

$$\mathbf{D}^b(\text{coh}(Y)) \cong \mathbf{D}^b(\text{mod } -B_\Sigma),$$

*where  $B_\Sigma$  is the algebra of endomorphisms of the exceptional collection  $\Sigma$  (3).*

## 4. Exceptional collections on Burniat surfaces

We begin with the following numerical statement:

**Lemma 4.1.** *Let  $\bar{L}_1, \bar{L}_2$  be two line bundles on  $Y$  and  $L_1, L_2$  be any line bundles on  $X$  lifting them under the projection  $\text{Pic } X \rightarrow \text{Pic } X / \text{Tors} = \text{Pic } Y$ . Then one has  $\chi(X, L_1 \otimes L_2^{-1}) = \chi(Y, \bar{L}_2 \otimes \bar{L}_1^{-1})$*

*Proof.* Denoting  $\mathcal{O}_Y(\bar{D}) = \bar{L}_1 \otimes \bar{L}_2^{-1}$  and  $\mathcal{O}_X(D) = L_1 \otimes L_2^{-1}$ , by Riemann-Roch:

$$\chi(X, D) = \frac{D(D - K_X)}{2} = \frac{\bar{D}(\bar{D} + K_Y)}{2} = \frac{-\bar{D}(-\bar{D} - K_Y)}{2} = \chi(Y, -\bar{D}).$$

□

**Corollary 4.2.** *For any exceptional sequence  $(\bar{L}_1, \dots, \bar{L}_n)$  of line bundles on  $Y$ ,  $(L_n, \dots, L_1)$  is a numerical exceptional sequence on  $X$ , i.e.  $\chi(L_i \otimes L_j^{-1}) = 0$  for  $i > j$ . If in addition  $H^0(X, L_i \otimes L_j^{-1}) = H^0(X, L_i \otimes L_j^{-1}) = 0$  for  $i > j$  then also  $H^1(X, L_i \otimes L_j^{-1}) = 0$  and  $L_n, \dots, L_1$  is an exceptional collection on  $X$ .*

For our example, we start with the exceptional collection on  $Y$  of the previous section and lift it to an exceptional collection on  $X$  by checking that for the corresponding differences  $\mathcal{O}(D) = L_i \otimes L_j^{-1}$  one has  $h^0(D) = 0$  and  $h^2(D) = h^0(K_X - D) = 0$ . Note that for all the differences  $D$  involved, both  $D$  and  $K_X - D$  are of the form (effective) + (torsion). Their images in  $\text{Pic } Y = \text{Pic } X / \text{Tors}$  are effective. However, since the torsion group of  $\text{Pic } X$  is so large, it is possible that  $D$  and  $K_X - D$  themselves are not effective for a wise choice of the lifts  $L_i$ .



**Theorem 4.3.** *The sequences of line bundles  $\langle L_1, L_2, L_3, L_4, L_5, L_6 \rangle$  and  $\langle L_1, L_2, L_3, L_4, L_5, L'_6 \rangle$  given in Table 2 form exceptional sequences on  $X$  that split into three blocks of sizes  $2 + 3 + 1$ , and the sheaves in the same block are mutually orthogonal.*

	$d$	$a_0$	$b_0$	$c_0$	$a_3$	$b_3$	$c_3$
$L_1$	3	0 00	0 00	0 00	1 10	1 10	1 10
$L_2$	3	1 10	1 10	1 10	0 00	0 00	0 00
$L_3$	2	1 11	0 01	0 00	1 11	0 01	0 00
$L_4$	2	0 01	0 00	1 11	0 01	0 00	1 11
$L_5$	2	0 00	1 11	0 01	0 00	1 11	0 01
$L_6$	0	0 00	0 00	0 00	0 00	0 00	0 00
$L'_6$	0	0 10	0 10	0 10	0 10	0 10	0 10

TABLE 2. Two exceptional collections on a Burniat surface

**Remark 4.4.** In terms of the generators, these line bundles can be written as follows. We put  $L_i = \mathcal{O}_X(R_i)$  for some divisors  $R_i$ , and list  $R_i$ .

$$\begin{aligned}
R_1 &= A_3 + B_0 + C_0 + A_1 - A_2, \\
R_2 &= A_0 + B_3 + C_3 + A_2 - A_1, \\
R_3 &= C_2 + A_2 - C_0 - A_3 = C_1 + A_1 - C_3 - A_0, \\
R_4 &= B_2 + C_2 - B_0 - C_3 = B_1 + C_1 - B_3 - C_0, \\
R_5 &= A_2 + B_2 - A_0 - B_3 = A_1 + B_1 - A_3 - B_0, \\
R_6 &= 0, \\
R'_6 &= A_0 + B_0 + C_0 + A_3 + B_3 + C_3 - K_X = -R'_6.
\end{aligned}$$

One also has

$$R_1 + R_2 = R_3 + R_4 + R_5 = A_0 + B_0 + C_0 + A_3 + B_3 + C_3$$

*Proof.* We should check that for every difference  $D = R_j - R_i$ ,  $i < j$ , and also for all the differences in the same block one has  $h^0(D) = h^0(K_X - D) = 0$ . The task is made easier by the  $\mathbb{Z}_3 \times \mathbb{Z}_2 = \mathbb{Z}_6$  symmetry group of this collection and of the Burniat configuration. Almost all the cases are handled by the following two elementary considerations:

- (1) If  $DK < 0$  or  $DK = 0$  but  $D \neq 0$  then  $D$  is not effective.
- (2) If for one of the elliptic curves  $E = A_0, \dots, C_3$  one has  $DE = 0$  and  $D|_E \neq 0$  in  $E[2]$  then  $E$  is in the base locus of the linear system  $|D|$ . Thus, if  $D$  is effective then so is  $D - E$ .

The remaining cases are handled by Lemma 4.5. We now go through the computation.

$R_5 - R_6$  has  $C_0, C_3$  in base locus.  $(R_5 - R_6) - C_0 - C_3$  has degree 0 but is not zero itself in  $\text{Pic } X$ . Done.

Below, we will abbreviate this sequence of arguments as follows: “ $R_5 - R_6 \rightarrow D - C_0 C_3$ .  $DK = 0$  but  $D \neq 0$ .” The symbol  $D$  will stand for the divisor discussed immediately before, e.g. for the above example we first have  $D = R_5 - R_6$  and then  $D = R_5 - R_6 - C_0 - C_3$ .

$K - (R_5 - R_6) \rightarrow D - B_0 B_3 \rightarrow D - C_0 C_3$ .  $DK = 0$  but  $D \neq 0$ .

$R_4 - R_6, R_3 - R_6$  differ from  $R_5 - R_6$  by a  $\mathbb{Z}_3$  symmetry.

$R_4 - R_5$ .  $DK = 0$  but  $D \neq 0$ .

$K - (R_4 - R_5) \rightarrow D - C_0C_3 \rightarrow D - A_0A_3 \rightarrow D - B_0B_3C_0C_3$ .  $DK < 0$ .

$R_5 - R_4$ .  $DK = 0$  but  $D \neq 0$ .

$K - (R_5 - R_4) \rightarrow D - B_0B_3 \rightarrow D - A_0A_3 \rightarrow D - B_0B_3$ .  $DK = 0$  but  $D \neq 0$ .

Other  $R_i - R_j, i, j \in \{3, 4, 5\}$  are done by symmetry.

$R_2 - R_6$  is done by Lemma 4.5.  $R_1 - R_6$  differs from it by a  $\mathbb{Z}_2$  symmetry.

$K - (R_2 - R_6) \rightarrow D - A_0B_0C_0$ .  $DK = 0$  but  $D \neq 0$ .

$R_2 - R_5 \rightarrow D - B_0C_3$ .  $DK < 0$ .

$K - (R_2 - R_5) \rightarrow D - A_0C_0 \rightarrow D - A_3B_3C_3$ .  $DK = 0$  but  $D \neq 0$ .

$R_2 - R_4, R_2 - R_3$  differ from  $R_2 - R_5$  by a  $\mathbb{Z}_3$  symmetry.

$R_1 - R_i, i \in \{4, 5, 6\}$  differ from  $R_2 - R_i$  by a  $\mathbb{Z}_2$  symmetry.

$R_1 - R_2$ .  $DK = 0$  but  $D \neq 0$ .

$K - (R_1 - R_2) \rightarrow D - A_3B_3C_3 \rightarrow D = R_1$ . Done by Lemma 4.5.

$R_2 - R_1$  is symmetric to  $R_1 - R_2$ .

The differences involving  $R'_6$ , up to symmetry:

$R_5 - R'_6 \rightarrow D - A_0C_0A_3C_3$ .  $DK < 0$ .

$K - (R_5 - R'_6) \rightarrow D - B_0B_3 \rightarrow D - A_0C_0A_3C_3$ .  $DK < 0$ .

$R_2 - R'_6 \rightarrow D - A_3B_3C_3$ .  $DK = 0$  but  $D \neq 0$ .

$K - (R_2 - R'_6) \rightarrow D = R_2$ . Done by Lemma 4.5.

This concludes the proof of the theorem.  $\square$

**Lemma 4.5.** *The divisor  $R_2$  is not effective.*

*Proof.* Consider the “corner”  $A_0 \cap C_3$  of the hexagon in Figure 1. We claim that any effective divisor  $D \in |R_2|$  contains either  $A_0$  or  $C_3$ . Indeed,  $R_2A_0 = 1$ . So either  $A_0 \leq D$  or  $D$  intersects  $A_0$  at a unique point giving (1 10) in  $\text{Pic } A_0$  in our coordinates, which is precisely  $A_0 \cap C_3$ . But since  $R_2C_3 = 0$ ,  $D$  must contain  $C_3$ .

The same argument applies to the “corners”  $B_0 \cap A_3$  and  $C_0 \cap B_3$ . Each of the six curves  $A_0, \dots, C_3$  passes through only one of them. Thus,  $D$  must contain at least three of the curves  $A_0, \dots, C_3$ . For degree reasons,  $D$  must be equal to the sum of exactly three of them. By focusing on the coordinates in  $A_0[2], B_0[2], C_0[2]$ , it follows that one must have  $D = A_3 + B_3 + C_3$ . But  $L_2A_3 = 0$  and  $(A_3 + B_3 + C_3)A_3 = -1$ . Contradiction.  $\square$

Consider the exceptional collection  $\Upsilon = (L_1, \dots, L_6 \cong \mathcal{O}_X)$  and denote by  $\mathcal{D}$  the full triangulated subcategory of  $\mathbf{D}^b(\text{coh}(X))$  generated by this collection. The subcategory  $\mathcal{D}$  is admissible, i.e. the embedding functor  $j : \mathcal{D} \rightarrow \mathbf{D}^b(\text{coh}(X))$  has right and left adjoint functors (see [BK89]).

Let us calculate the DG algebra of endomorphisms  $\mathcal{B}_\Upsilon = \mathbf{R}\text{Hom}(T, T)$ , where  $T = \bigoplus_{i=1}^6 L_i$ . First, we should find all nontrivial  $\text{Hom}$ ’s and  $\text{Ext}$ ’s between line bundles  $L_i$  and  $L_j$  in our collection.

**Lemma 4.6.** *For the exceptional collection  $\Upsilon = (L_1, \dots, L_6)$  we have*

$$\text{Hom}(L_i, L_j) = 0 \quad \text{for all } i \neq j.$$

*The same holds for the exceptional collection  $\Upsilon' = (L_1, \dots, L'_6)$ .*

*Proof.* Since the collection  $\Upsilon$  is exceptional, it remains to check only the case when  $i < j$  and  $L_i, L_j$  are from different blocks. But in this case  $\text{Hom}(L_i, L_j) \cong H^0(X, \mathcal{O}(R_j - R_i)) = 0$ , because  $(R_j - R_i)K < 0$ .  $\square$

**Lemma 4.7** ([MLP01] Lemma 5.6(3)). *If  $h^1(X, \mathcal{O}(\tau)) > 0$  for a torsion divisor  $\tau \in \text{Tors}(\text{Pic } X)$  then the restrictions of  $\tau$  on  $A_0, B_0$ , and  $C_0$  are trivial for two of them and is equal to (10) for the third. Hence, for all other nonzero torsion divisors  $\alpha$  we have  $h^1(X, \mathcal{O}(\alpha)) = 0$  and  $h^2(X, \mathcal{O}(\alpha)) = \chi(\mathcal{O}(\alpha)) = \chi(\mathcal{O}) = 1$ .*

**Lemma 4.8.** *For the exceptional collection  $\Upsilon = (L_1, \dots, L_6)$  we have*

$$\text{Ext}^1(L_i, L_j) = 0 \quad \text{for all } i, j.$$

*The same results hold after replacing  $L_6$  by  $L'_6$ .*

*Proof.* Since the collection  $\Upsilon$  is exceptional, it remains to check only the cases when  $i < j$  and  $L_i, L_j$  are from different blocks.

Consider two line bundles  $L_1$  and  $L_3$  from the first and the second blocks. The line bundle  $L_3 \otimes L_1^{-1}$  is isomorphic to  $\mathcal{O}_X(-A_0 + \alpha)$ , where  $\alpha$  is a torsion divisor. It follows from Lemma 4.7 and Table 2 that  $h^1(X, \mathcal{O}(\alpha)) = 0$  and the restriction  $\alpha|_{A_0}$  is not trivial. Therefore  $h^0(A_0, \alpha|_{A_0}) = 0$  and from the short exact sequence

$$0 \longrightarrow \mathcal{O}(-A_0 + \alpha) \longrightarrow \mathcal{O}(\alpha) \longrightarrow \mathcal{O}_{A_0}(\alpha) \longrightarrow 0$$

we obtain that  $H^1(X, \mathcal{O}(-A_0 + \alpha)) = 0$ . Similar considerations work for all line bundles from the first and the second blocks. We only have to replace  $A_0$  by another elliptic curve  $B_0, C_0, A_3, B_3, C_3$ , respectively.

Now consider  $L_3$  and  $L_6 = \mathcal{O}_X$ . The line bundle  $L_3^{-1}$  is isomorphic to  $\mathcal{O}_X(-B_0 - C_3 + \beta)$ , where  $\beta$  is a torsion divisor. Consider the following short exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}(-B_0 + \beta) \longrightarrow \mathcal{O}(\beta) \longrightarrow \mathcal{O}_{B_0}(\beta) \longrightarrow 0 \\ 0 &\longrightarrow \mathcal{O}(-B_0 - C_3 + \beta) \longrightarrow \mathcal{O}(-B_0 + \beta) \longrightarrow \mathcal{O}_{C_3}(-B_0 + \beta) \longrightarrow 0, \end{aligned}$$

It follows from Lemma 4.7 and Table 2 that  $h^1(X, \mathcal{O}(\beta)) = 0$  and the restrictions  $\beta|_{B_0}$  is not trivial. Therefore  $h^1(X, \mathcal{O}(-B_0 + \beta)) = 0$ . From the second exact sequence, noting that  $\deg \mathcal{O}_{C_3}(-B_0 + \beta) < 0$ , we obtain  $h^1(X, \mathcal{O}(-B_0 - C_3 + \beta)) = 0$ . Similar considerations work for any line bundle  $L_i, i = 3, 4, 5$  from the second block and  $L_6 = \mathcal{O}_X$ .

Finally, let us take  $L_1$  and  $L_6 = \mathcal{O}_X$ . The line bundle  $L_1^{-1}$  is isomorphic to  $\mathcal{O}_X(-A_0 - C_1 + \gamma)$ , where  $\gamma$  is a torsion divisor. Consider the following short exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathcal{O}(-A_0 + \gamma) \longrightarrow \mathcal{O}(\gamma) \longrightarrow \mathcal{O}_{A_0}(\gamma) \longrightarrow 0 \\ 0 &\longrightarrow \mathcal{O}(-A_0 - C_1 + \gamma) \longrightarrow \mathcal{O}(-A_0 + \gamma) \longrightarrow \mathcal{O}_{C_1}(-A_0 + \gamma) \longrightarrow 0 \end{aligned}$$

It follows from Lemma 4.7 and Table 2 that  $h^1(X, \mathcal{O}(\gamma)) = 0$  and the restriction  $\gamma|_{A_0}$  is not trivial. Therefore  $h^1(X, \mathcal{O}(-A_0 + \gamma)) = 0$ . From the second exact sequence, noting that  $\deg \mathcal{O}_{C_1}(-A_0 + \gamma) < 0$ , we obtain that  $h^1(X, \mathcal{O}(-A_0 - C_1 + \gamma)) = 0$ .

The case of the line bundles  $L_2$  and  $L_6$  works by symmetry. The Ext groups involving  $L'_6$  are handled the same way.  $\square$

Lemmas 4.6 and 4.8 and calculation of Euler characteristic immediately imply the following proposition.

**Proposition 4.9.** *For the exceptional collection  $\Upsilon = (L_1, \dots, L_6)$  the nontrivial Ext groups are the following*

- 1)  $\text{Ext}^2(L_i, L_j) \cong k$  for  $i = 1, 2; j = 3, 4, 5$
- 2)  $\text{Ext}^2(L_j, L_6) \cong k^2$  for  $j = 3, 4, 5$
- 3)  $\text{Ext}^2(L_i, L_6) \cong k^3$  for  $i = 1, 2$ .

*The same results hold after replacing  $L_6$  by  $L'_6$ .*

Let us consider the DG algebra  $\mathcal{B}_\Upsilon = \mathbf{R}\text{Hom}(T, T)$ , where  $T = \bigoplus_{i=1}^6 L_i$ . The minimal model  $H^*(\mathcal{B}_\Upsilon)$  of this DG algebra is an  $A_\infty$ -algebra that by Proposition 4.9 has only two nontrivial graded terms

$$H^0(\mathcal{B}_\Upsilon) \cong \bigoplus_{i,j} \text{Hom}(L_i, L_j) \cong k^6, \quad \text{and} \quad H^2(\mathcal{B}_\Upsilon) = \bigoplus_{i,j} \text{Ext}^2(L_i, L_j).$$

This  $A_\infty$ -algebra is actually an  $A_\infty$ -category on our 6 objects  $L_1, \dots, L_6$ . The  $H^0(\mathcal{B}_\Upsilon)$  is the sum of identity morphisms of these 6 objects. By standard theorems (see [Sei08] Lemma 2.1 or [Lef02] Th 3.2.1.1), any such  $A_\infty$ -category is equivalent to a strict  $A_\infty$ -category. This means we can assume that  $m_l(\dots, \text{id}_{L_i}, \dots) = 0$  for all objects  $L_i$  and all  $l > 2$ .

Thus all nontrivial  $m_l$  for  $l > 2$  can exist only if all elements are from  $H^2(\mathcal{B}_\Upsilon)$ . But they are also trivial for dimension reasons: the degree of  $m_l$  is  $2 - l$ , so  $m_l(a_1, \dots, a_k) \in H^{l+2}(\mathcal{B}_\Upsilon) = 0$ . Thus, one has  $m_l = 0$  for all  $l > 2$ . This means that the DG algebra  $\mathcal{B}_\Upsilon$  and the  $A_\infty$ -algebra  $H^*(\mathcal{B}_\Upsilon)$  are formal and are quasi-isomorphic to a usual graded algebra. Moreover, all compositions of all elements of degree 2 are 0. Thus, the algebra  $H^*(\mathcal{B}_\Upsilon)$  is not changed under a deformation of a Burniat surface  $X$ . Thus, we obtain the following:

**Proposition 4.10.** *Let  $\Upsilon = (L_1, \dots, L_6)$  be the exceptional collection constructed above. Then the DG algebra  $\mathcal{B}_\Upsilon = \mathbf{R}\text{Hom}(T, T)$ , where  $T = \bigoplus_{i=1}^6 L_i$ , of endomorphism of this collection is formal, i.e. it is quasi-isomorphic to its cohomology algebra  $H^*(\mathcal{B}_\Upsilon)$ . The admissible subcategory  $\mathcal{D} \subset \mathbf{D}^b(\text{coh}(X))$  generated by  $\Upsilon = (L_1, \dots, L_6)$  is the same for all Burniat surfaces  $X$ , and  $\mathcal{D} \cong \text{Perf}(H^*(\mathcal{B}_\Upsilon))$ .*

**Remark 4.11.** Note that a Burniat surface  $X$  can be reconstructed from the bounded derived category of coherent sheaves  $\mathbf{D}^b(\text{coh}(X))$ , because the canonical class  $K_X$  is ample (see [BO01]).

As it was mentioned above, the subcategory  $\mathcal{D}$  generated by an exceptional collection is admissible, i.e. the embedding functor has right and left adjoint functors. Thus we obtain a semiorthogonal decomposition

$$\mathbf{D}^b(\text{coh}(X)) = \langle \mathcal{D}, \mathcal{A} \rangle$$

where  $\mathcal{A}$  is a left orthogonal to  $\mathcal{D}$ , i.e.  $\mathcal{A}$  consists of all object  $A$  such that  $\text{Hom}(A, D) = 0$  for all objects  $D \in \mathcal{D}$ .

A semiorthogonal decomposition implies direct sum decompositions for K-theory and Hochschild homology (see, for example, [Kuz09] for Hochschild homology). Thus, we obtain

$$K_0(X) = K_0(\mathcal{D}) \oplus K_0(\mathcal{A}), \quad \text{and} \quad \text{HH}_*(X) = \text{HH}_*(\mathcal{D}) \oplus \text{HH}_*(\mathcal{A})$$

All these invariants are defined for  $\mathcal{D}$  and  $\mathcal{A}$ , because they have induced DG enhancements.

There is an isomorphism for Hochschild homology of a smooth projective variety

$$\mathrm{HH}_i(X) = \bigoplus_p H^{p+i}(X, \Omega^p X)$$

For a Burniat surface it gives that  $\mathrm{HH}_i(X) = 0$  when  $i \neq 0$ , and  $\mathrm{HH}_0(X) \cong k^6$ . The subcategory  $\mathcal{D}$  as a subcategory generated by an exceptional collection of length 6 has the same Hochschild homology. Hence, we obtain that  $\mathrm{HH}_*(\mathcal{A}) = 0$ .

Recall that Bloch conjecture for Burniat surfaces was proved by Inose and Mizukami in [IM79], i.e. the Chow group  $\mathrm{CH}^2(X)$  of 0-cycles is isomorphic to  $\mathbb{Z}$ . Therefore, for  $K$ -theory we obtain that  $K_0(X) \cong \mathbb{Z}^6 \oplus \mathbb{Z}_2^6$ . It is evident that  $K_0(\mathcal{D}) \cong \mathbb{Z}^6$ . Hence,  $K_0(\mathcal{A}) \cong \mathbb{Z}_2^6$ .

Let us summarize what we have just established.

**Theorem 4.12.** *For any Burniat surface  $X$  we have a semiorthogonal decomposition*

$$\mathbf{D}^b(\mathrm{coh}(X)) = \langle \mathcal{D}, \mathcal{A} \rangle,$$

where  $\mathcal{D}$  is a subcategory generated by an exceptional collection  $\Upsilon = (L_1, \dots, L_6)$ . The category  $\mathcal{D}$  is the same for all Burniat surfaces. The category  $\mathcal{A}$  has trivial Hochschild homology and  $K_0(\mathcal{A}) = \mathbb{Z}_2^6$ . The same holds after replacing  $L_6$  by  $L'_6$ .

## 5. Further remarks

It is proved in [Orl09] that for any quasi-projective scheme  $Z$  of dimension  $d$  and a very ample line bundle  $L$ , the object  $\mathcal{E} = \bigoplus_{i=0}^d L^i$  is a classical generator for the triangulated category of perfect complexes  $\mathrm{Perf}(Z)$ . This means that  $\mathcal{E}$  idempotently generates the category  $\mathrm{Perf}(Z)$ , i.e. the minimal idempotent complete triangulated subcategory containing it coincides with  $\mathrm{Perf}(Z)$  (see [Orl09] for details).

This result can be easily made a little stronger: it is sufficient to assume that  $L = f^*L'$  is a pull back of a very ample line bundle  $L'$  under a finite morphism  $f : Z \rightarrow W$  (i.e. that  $L$  is an ample and semiample line bundle). Indeed, since  $L'$  is very ample on  $W$ , the object  $\mathcal{E}' = \bigoplus_{i=0}^d (L')^i$  idempotently generates the category  $\mathrm{Perf}(W)$ . Hence, all  $L^m = f^*(L')^m$  belong to the idempotent complete subcategory generated by  $\mathcal{E} = f^*\mathcal{E}'$ . Some power of  $L$  is very ample, so the minimal derived category containing  $\mathcal{E} = \bigoplus_{i=0}^d L^i$  is the whole  $\mathrm{Perf}(Z)$ .

On a Burniat surface,  $\mathcal{O}_X(2K_X) = f^*\mathcal{O}_Y(-K_Y)$  is ample and semiample. Thus,  $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X(2K_X) \oplus \mathcal{O}_X(4K_X)$  is a classical generator for  $\mathrm{Perf}(X) = \mathbf{D}^b(\mathrm{coh}(X))$ .

By Keller's results [Kel94], this means that the bounded derived category  $\mathbf{D}^b(\mathrm{coh}(X))$  is equivalent to the triangulated category of perfect objects  $\mathrm{Perf}(\mathcal{B}_{\mathcal{E}})$  over the DG algebra of endomorphisms  $\mathcal{B}_{\mathcal{E}} = \mathbf{R}\mathrm{Hom}(\mathcal{E}, \mathcal{E})$ . Furthermore, we can try to produce a classical generator for the full admissible subcategory  $\mathcal{A}$  considering a projection of  $\mathcal{E}$  to  $\mathcal{A}$ . Since  $\mathcal{O}_X$  belongs to  $\mathcal{D}$ , its projection is trivial and we have to take in account only projections of  $\mathcal{O}_X(2K_X)$  and  $\mathcal{O}_X(4K_X)$  to  $\mathcal{A}$ . It will be very interesting to consider this generator for the subcategory  $\mathcal{A}$  and to calculate the DG (or  $A_{\infty}$ ) algebra of endomorphisms for it.

One can speculate whether our results are but one example of a general phenomenon.

**Question 5.1.** Is it true that for any exceptional collection  $\Upsilon$  of maximal length on a smooth surface  $X$  with ample  $K_X$  and with  $p_g = q = 0$ , the DG algebra of endomorphisms of  $\Upsilon$  does not change under small deformations of the complex structure on  $X$ ?

Certainly, if the surface  $X$  is a “fake del Pezzo surface” and the exceptional collection  $\Upsilon = (L_n, \dots, L_1)$  lifts a nice enough exceptional collection  $(\bar{L}_1, \dots, \bar{L}_n)$  on the corresponding del Pezzo surface  $Y$ , then as in Lemma 4.6 it follows that  $\text{Hom}(\bar{L}_i, \bar{L}_j) = 0$ . If one is lucky and  $\text{Ext}^1(L_i, L_j) = 0$  as well then, as we explained on page 12, the DG algebra of endomorphisms of  $\Upsilon$  is formal and, moreover, it does not have any deformations.

But even if we are not in such a “lucky” situation, the general take-away from our computations seems to be that under the correspondence between a true del Pezzo surface  $Y$  and its “fake partner”  $X$  that switches  $-K_Y$  and  $K_X$ , the groups  $\text{Ext}^0(\bar{L}_j, \bar{L}_i)$ ,  $i \neq j$ , switch their places with  $\text{Ext}^2(L_i, L_j)$ . On a del Pezzo, the Hom groups for *some* exceptional collection completely describe  $\mathbf{D}^b(\text{coh}(Y))$  and thus also the position of  $Y$  in the moduli space. But on the surface of general type, all this information is pushed to the  $\text{Ext}^2$ ’s which for degree reasons lead to a rigid DG algebra and say *nothing* about the position of  $X$  in the moduli space.

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